Department of Mathematics

Comprehensive Examination–Option I

2016 Spring

Algebra

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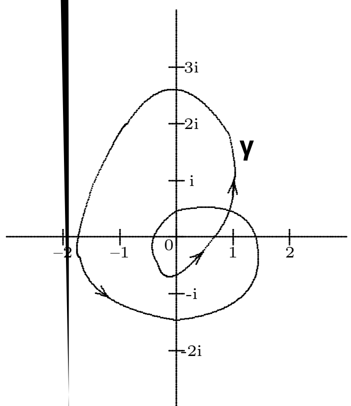
Complex Analysis

1. Evaluate

$$\int_\gamma \frac{(z-1)\,dz}{z(z^2+9)(z+1)^3}$$

where γ is the contour illustrated.



2. Find a0R159.96264 Tf10.56350.24 Td[(82]TJ/R15(37R135.96264 Tf-241.2-93.84 Td[(2)-5.89017(.)-493.09735m[(:)17.3382]T

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Real Analysis

1. (a) Complete the $\varepsilon - N$ definition:

A sequence (x_n) of real numbers converges to a limit

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Topology

1. Let A be a subset of a topological space X. The boundary of A, $\partial A = \overline{A} \cap \overline{A^c}$, is the intersection of the closure of A and the closure of its complement.
   (a) Prove that ∂A and A° are disjoint, where A° is the interior of A.
   (b) Prove that $\overline{A} = \partial A \cup A^\circ$.

2. Let X be an infinite topological space with co-finite topology; *i.e.*, $A \subset X$ is open iff $A = \emptyset$ or $X - A$ is a finite set.
   (a) Prove that X is not Hausdorff.
   (b) Prove that X is connected.

3. Prove that the continuous image of a compact topological space is compact, and then prove that if X and Y are topological spaces such that $X \times Y$ is compact, then each of X and Y is compact.

4. If A is a subset of a topological space X, then A is a <u>retract</u> of X if there exists a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$

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Applied Analysis

1. Find the first three nonzero terms of two linearly independent power series solutions based at the origin

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Numerical Analysis

1. (a) Prove that there exist exactly two real solutions, one negative and one positive, of the equation $e^x = 3x + 2$.
   (b) Find an approximation β of the positive solution α such that $|\alpha - \beta| < 10^{-6}$.
   (c) Prove that your approximation is in fact within 10^{-6} of (the exact) α.

   Note: For this problem you may not use any graphing or root finding capabilities of your calculator.

2. Suppose $f : [0,1] \to \mathbb{R}$ by $f(x) = e^{-x}$.
   (a) Find the polynomial $p(x)$ such that degree$(p(x)) \le 2$, $p(0) = f(0)$, $p(1/2) = f(1/2)$, and $p(1) = f(1)$.
   (b) Using the formula from the interpolating polynomial error theorem show that
      i. $e^{-1/3} - p(1/3) < 0$, and
      ii. $|e^{-1/3} - p(1/3)| < 1/162 < 0.00625$.

3. Let
$$A = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}.$$
   It can be shown that A is positive definite.
   (a) Construct T

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Linear Programming

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Linear Programming–continued

3. Using the Complementary Slackness Theorem prove or disprove the following statement. $[3, 0, 5]$ is the optimal solution of the maximization problem below.

$$
\begin{array}{rrcrcrcl}
\text{maximize} & 9x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{subject to} & 3x_1 & - & x_2 & - & 6x_3 & \leq & 26 \\
 & 6x_1 & + & 3x_2 & + & 2x_3 & \leq & 28 \\
 & -3x_1 & + & 2x_2 & + & 4x_3 & \leq & 11 \\
 & & & & & x_j & \geq & 03
\end{array}
$$

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Probability

1. MEGA Millions is a lottery game available to players who are 18 and older in certain states. In the game 5 white balls numbered from 1 to 75 are drawn from a drum without replacement. A sixth gold "MEGA Ball" (numbered 1 to 15) is drawn from a separate drum. The MEGA Ball is considered separate from the white balls.
   (a) What is the probability that someone matches all 5 white numbers and the gold MEGA Ball number?
   (b) What is the probability that someone matches all five white numbers?
   (c) Of the five white numbers drawn in the last drawing, what is the probability that none of these is drawn in this drawing?
   (d) Of the five white numbers drawn in the last drawing, what is the probability that at least one of those is drawn in this drawing?
   (e) What is the probability that a ticket has none of the 5 white numbers or the gold MEGA Ball number?

2. The length of time in hours needed to complete a task follows the probability density function defined below. Let X = time to completion.

$$f(x) = \begin{cases} cx^2 + x, & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find c.
   (b) Show that $f(x)$ is a probability density function.
   (c) Find the probability that the task can be completed in less than 1/2 hour.
   (d) Find the cumulative distribution function.
   (e) Given that Sally needs at least 15 minutes to complete the task, find the probability that Sally will take over 30 minutes to complete the task.

3. If heads is a success when we flip a coin, getting a six is a success when we roll a die, and getting an ace is a success when we draw a card from an ordinary deck of 52 playing cards, find the mean and standard deviation of the total number of successes when we
   (a) flip a fair coin, roll a balanced die, and then draw a card from a well-shuffled deck;
   (b) flip a fair coin three times, roll a balanced die twice, and then draw a card from a well-shuffled deck.

Probability–continued

4. Suppose three urns numbered 1, 2, and 3 contain, respectively, one red and one blue ball, two red and three blue balls, and four red and two blue balls. Consider an experiment consisting of the selection of an urn and followed by the draw of one ball from it. Define B_1, B_2, B_3 to be the event urn 1, 2, or 3 is selected; define A to be the event that a red ball is selected.

   (a) Assume equal probabilities for each urn selected:
   $P(B_1) = P(B_2) = P(B_3) = 1/3$.
   Compute the probability that a red ball is selected, $P(A)$.

   (b) If the urn number is not observed but a red ball is drawn, what is the probability that it was drawn from urn 1, urn 2, or urn 3? Compute $P(B_1|A)$, $P(B_2|A)$, and $P(B_3|A)$.

   (c) Verify that the conditional probabilities in part (b) are correct by computing the sum of the three probabilities. Which urn is most likely the one from which the red ball was drawn?

   (d) Now assume the probabilities for each urn are unequal:
   $P(B_1) = 1/2$, $P(B_2) = 1/3$, and $P(B_3) = 1/6$.
   Again, if the urn number is not observed but a red ball is drawn, what is the probability that it was drawn from urn 1, urn 2, or urn 3? Compute $P(B_1|A)$, $P(B_2|A)$, and $P(B_3|A)$. Which urn is most likely the one from which the red ball was drawn?